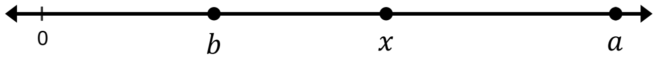
Mathematics, Music, and Means
Math Teacher's Circle
Tuesday, October 4, 2016

Warm-up

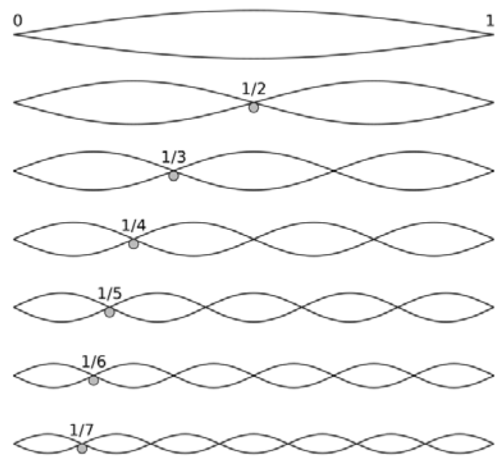
If $0 < b \leq x \leq a$ are the coordinates of points on the real number line, solve each of the following equations for x in terms of a and b.



1. $\frac{a-x}{x-b} = \frac{a}{a}$
2. $\frac{a-x}{x-b} = \frac{a}{x}$
3. $\frac{a-x}{x-b} = \frac{a}{b}$

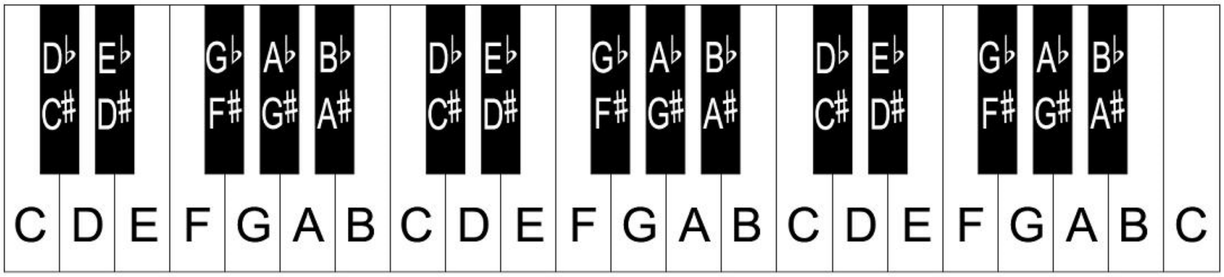
Just Intonation

Just Intonation is any musical tuning in which the frequencies of notes are related by ratios of small whole numbers. Any interval tuned in this way is called a pure or just interval. Pure intervals are important in music because they naturally tend to be perceived by humans as "consonant": pleasing or satisfying. Intervals not satisfying this criterion, conversely, tend to be perceived as unpleasant or as creating "dissonance": dissatisfaction or tension. The two notes in any just interval are members of the same harmonic series. Frequency ratios involving large integers such as 1024:729 are not generally said to be justly tuned. "Just intonation is the tuning system of the later ancient Greek modes as codified by Ptolemy; it was the aesthetic ideal of the Renaissance theorists; and it is the tuning practice of a great many musical cultures worldwide, both ancient and modern."



Equal Temperament

Just intonation can be contrasted and compared with equal temperament, which dominates Western instruments of fixed pitch (e.g., piano or organ) and default MIDI tuning on electronic keyboards. In equal temperament, all intervals are defined as multiples of the same basic interval, or more precisely, the intervals are ratios which are integer powers of the smallest step ratio, so two notes separated by the same number of steps always have exactly the same frequency ratio. However, except for doubling of frequencies (one or more octaves), no other intervals are exact ratios of small integers. Each just interval differs a different amount from its analogous, equally tempered interval.



Semitones	Solfège	Note	Just Intonation Scaling Factor	Frequency (Hz)	Equal Temperament Scaling Factor	Frequency (Hz)
0	Do	A	1/1	220	1	220
1		A#	16/15			
2	Re	B	9/8			
3		C	6/5			261.63
4	Mi	C#	5/4			
5	Fa	D	4/3			
6		D#	45/32	309.38	1.41421	
7	So	E	3/2	330		
8		F	8/5			
9	La	F#	5/3			
10		G	9/5			
11	Ti	G#	15/8			
12	Do	A	2/1	440	2	440

1. The expression for *x* in warm-up exercise #1 is known as the arithmetic mean (sometimes called the average) of *a* and *b*. Calculate the arithmetic mean of the notes A and E.

2. The expression for *x* in warm-up exercise #2 is known as the geometric mean of *a* and *b*. Calculate the geometric mean of the notes A and E.

3. The expression for *x* in warm-up exercise #3 is known as the harmonic mean of *a* and *b*. Calculate the harmonic mean of the notes A and E.

Solfège	Note
Do	C
Re	
Mi	
Fa	
So	
La	
Ti	
Do	

Solfège	Note
Do	E
Re	
Mi	
Fa	
So	
La	
Ti	
Do	

Solfège	Note
Do	G
Re	
Mi	
Fa	
So	
La	
Ti	
Do	

4. For 2 numbers $a < b$, Valerie says that you can compute the arithmetic mean with the formula $a + \frac{b-a}{2}$, but Ashley says she is wrong and that you have to use the formula $b - \frac{b-a}{2}$. Who is correct? Explain.

5. Write an expression for the arithmetic mean of three numbers, *a*, *b*, and *c*.

6. Could you calculate the arithmetic mean of 3 numbers by first averaging two numbers, and then averaging this result with the third number? Show why or why not.

7. To calculate the geometric mean of *n* numbers, multiply the numbers together and take the *n*th root of the result. Write an expression for the geometric mean of *a*, *b*, and *c*.

8. Could you calculate the geometric mean of 3 numbers by first finding the geometric mean of two numbers, and then finding the geometric mean of this result with the third number? Show why or why not.

9. One definition of the harmonic mean is "the reciprocal of the average of the reciprocals." Use this definition to write an expression for the harmonic mean of *a* and *b*. Once you have translated the expression, rewrite it as a simple fraction.

10. Use the definition for harmonic mean given in **9** to write an expression for the harmonic mean of *a*, *b*, and *c* rewrite it as a simple fraction.

11. Do the same (calculate the harmonic mean) for *a*, *b*, *c*, and *d*.

12. Can you make a conjecture about the harmonic mean of 5 numbers, *a*, *b*, *c*, *d*, and *e*?

13. Could you calculate the harmonic mean of 3 numbers by first finding the harmonic mean of two numbers, and then finding the harmonic mean of this result with the third number? Show why or why not.

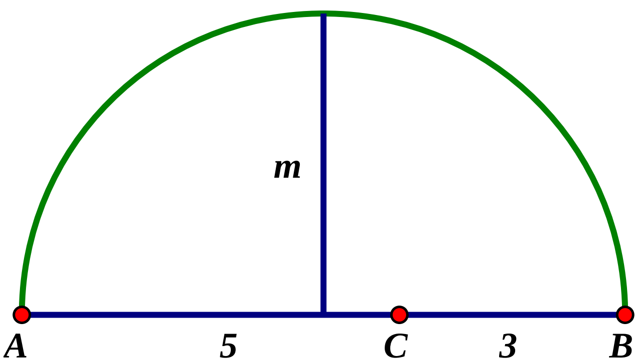
14. Find the arithmetic mean, geometric mean, and harmonic mean of 3 and 5.

15. Joe can paint a house in 3 hours. Sam can paint the same house in 5 hours. How long will it take them to paint the house together?

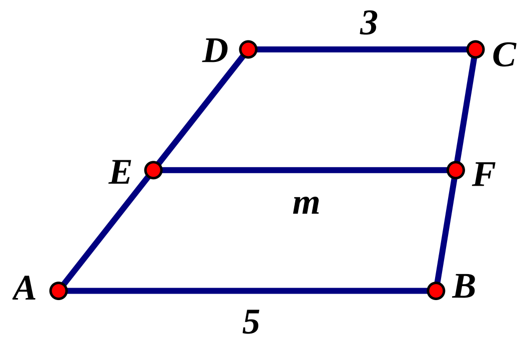
16. Joe can paint a house in *a* hours. Sam can paint the same house in *b* hours. How long will it take them to paint the house together?

17. How are questions 15 and 16 related to the harmonic mean?

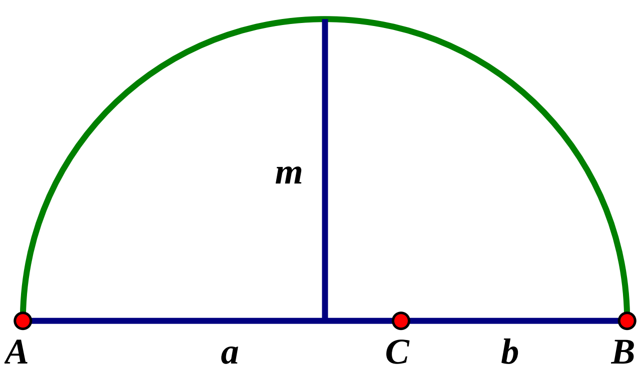
1. In the semicircle, segment *AC* of the diameter has length 5, and segment *BC* of the diameter has length 3. Find the length of the radius *m* of the semicircle.



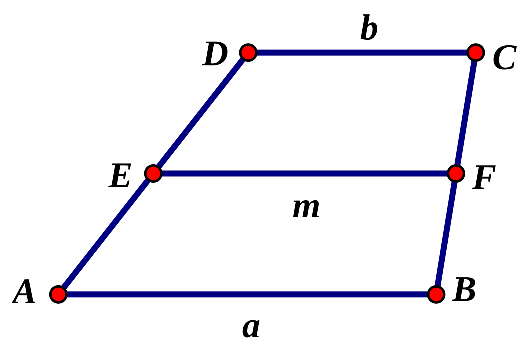
2. In trapezoid *ABCD*, base *AB* = 5 and base *CD* = 3. *E* is the midpoint of *AD*, and *F* is the midpoint of *BC*. Find the length *m* of segment *EF*.



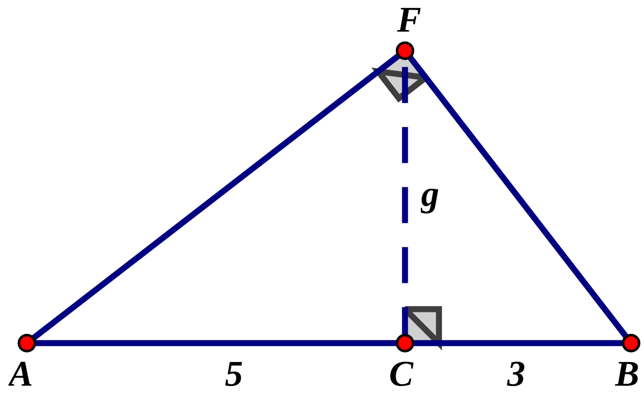
3. In the semicircle, segment *AC* of the diameter has length *a*, and segment *BC* of the diameter has length *b*. Find the length of the radius *m* of the semicircle in terms of *a* and *b*.



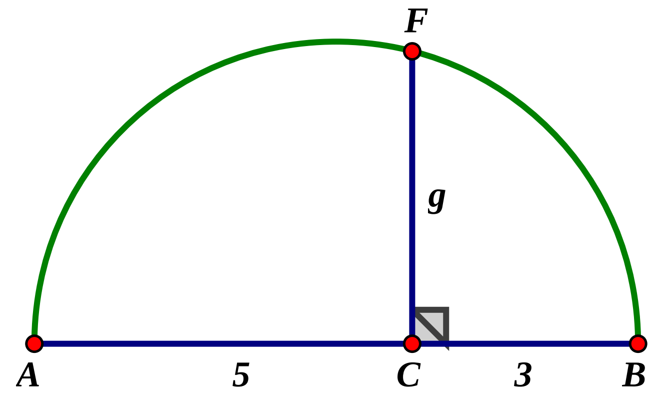
4. In trapezoid *ABCD*, base *AB* = *a* and base *CD* = b. *E* is the midpoint of *AD*, and *F* is the midpoint of *BC*. Find the length *m* of segment *EF* in terms of *a* and *b*.



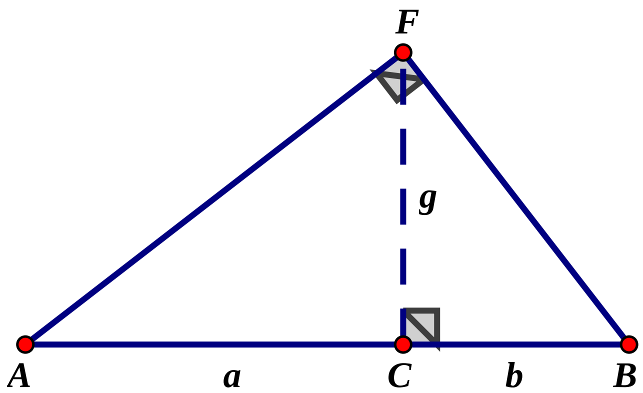
5. Find the length *g* of altitude *CF* in right triangle *ABF*.



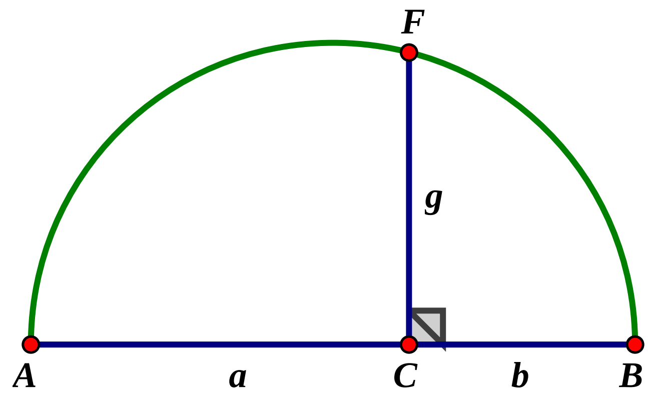
6. Find the length *g* of segment *CF*.



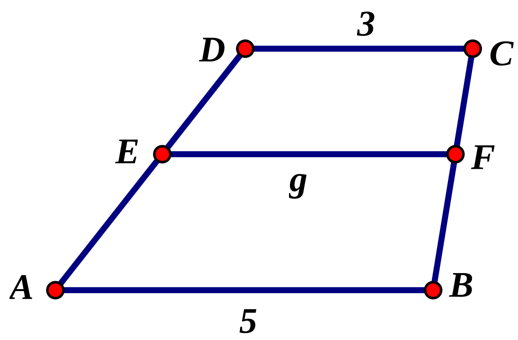
7. Find the length *g* of altitude *CF* in right triangle *ABF*, in terms of *a* and *b*.



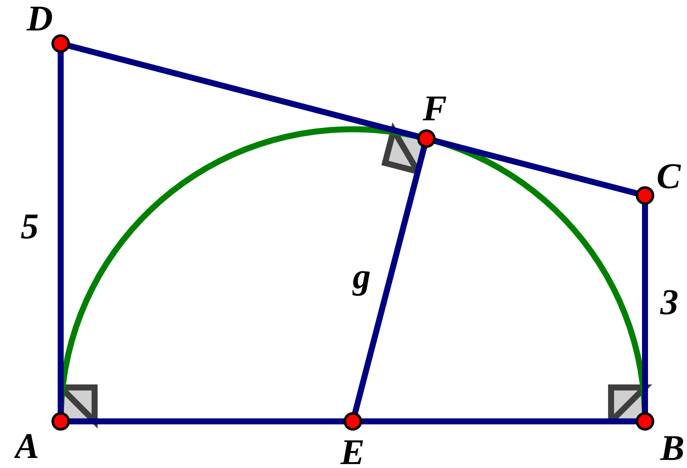
8. Find the length *g* of segment *CF*, in terms of *a* and *b*.



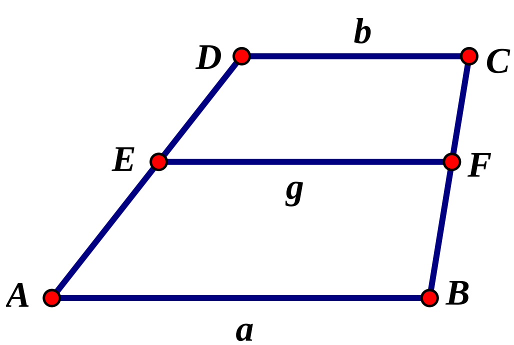
9. Trapezoid *ABFE* is similar to trapezoid *EFCD*. Find the length *g* of segment *EF*.



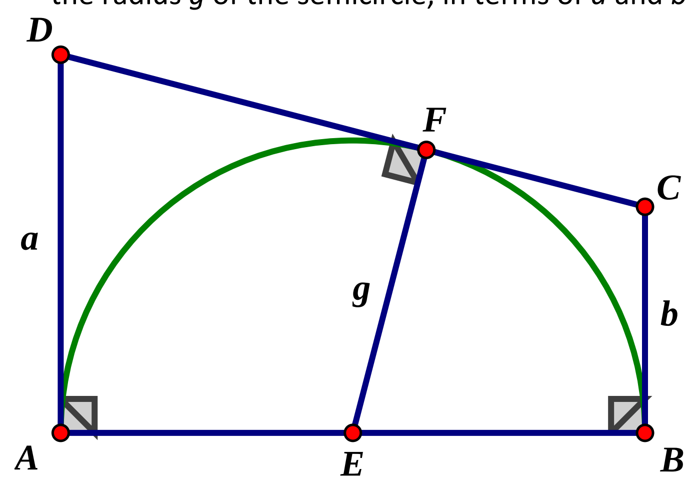
10. *AD*, *BC*, and *CD* are tangent to the semicircle. Find the radius *g* of the semicircle.



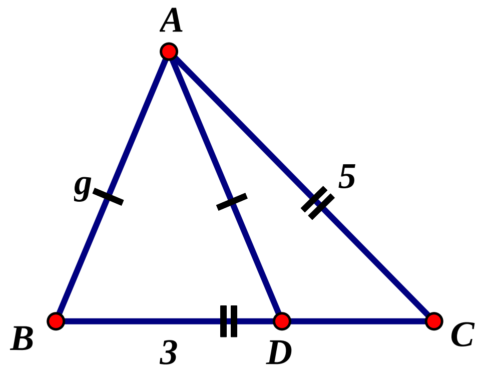
11. Trapezoid *ABFE* is similar to trapezoid *EFCD*. Find the length *g* of segment *EF*, in terms of *a* and *b*.



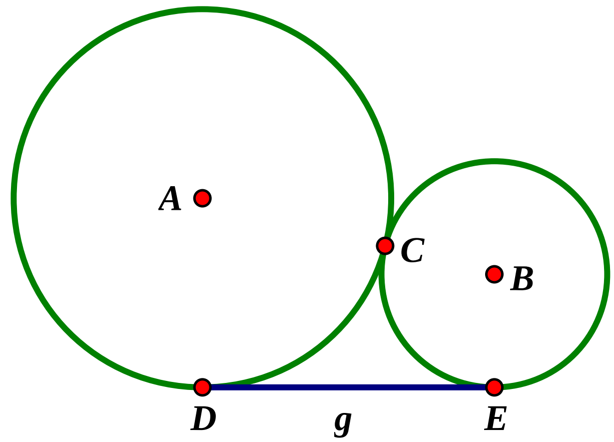
12. *AD*, *BC*, and *CD* are tangent to the semicircle. Find the radius *g* of the semicircle, in terms of *a* and *b*.



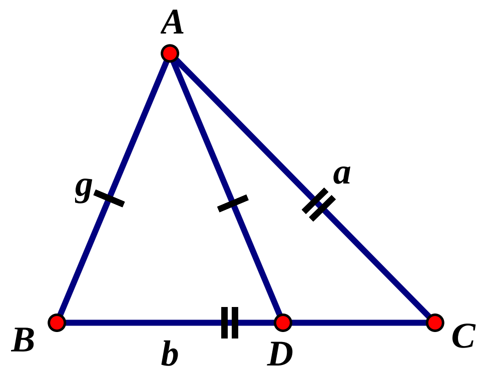
13. Isosceles triangle *ABC* is similar to isosceles triangle *BDA*. *AC* = 5 and *BD* = 3. Find the length of *g*.



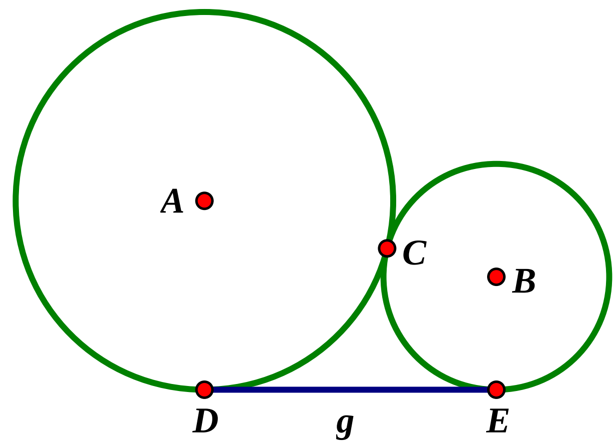
14. Circle *A* and circle *B* are externally tangent to each other at point *C*. The diameter of circle *A* is 5, and the diameter of circle *B* is 3. Find the length *g* of the common tangent *DE*.



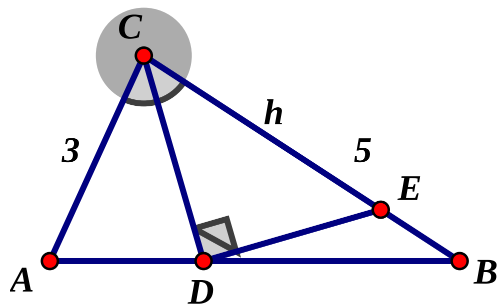
15. Isosceles triangle *ABC* is similar to isosceles triangle *BDA*. *AC* = *a* and *BD* = *b*. Find the length of *g*, in terms of *a* and *b*.



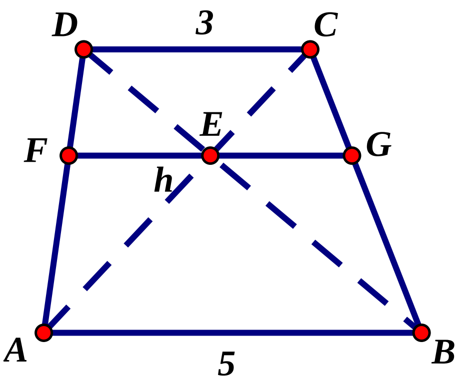
16. Circle *A* and circle *B* are externally tangent to each other at point *C*. The diameter of circle *A* is a, and the diameter of circle *B* is b. Find the length *g* of the common tangent *DE*, in terms of *a* and *b*.



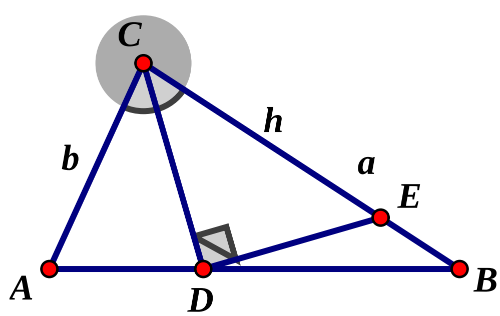
17. *AC* = 3 and *BC* = 5. *CD* bisects angle *C*. *DE* is perpendicular to *CD*. Find the distance *h* between points *C* and *E*.



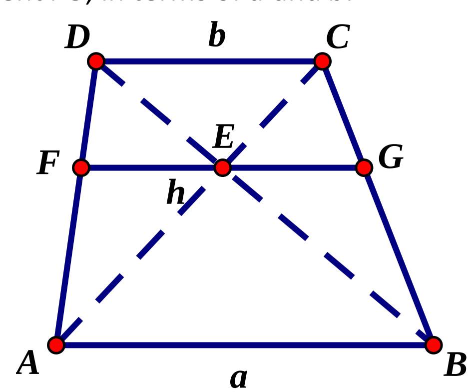
18. The diagonals of trapezoid *ABCD* intersect at *E*, and segment *FG* is parallel to the bases through *E*. *AB* = 5 and *CD* = 3. Find the length *h* of segment *FG*.



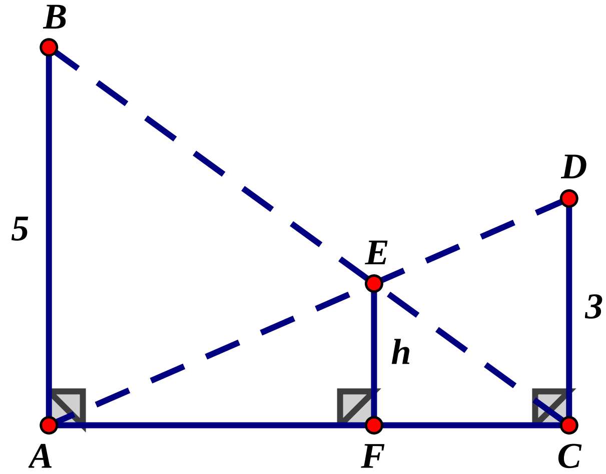
19. *AC* = *b* and *BC* = *a*. *CD* bisects angle *C*. *DE* is perpendicular to *CD*. Find the distance *h* between points *C* and *E*, in terms of *a* and *b*.



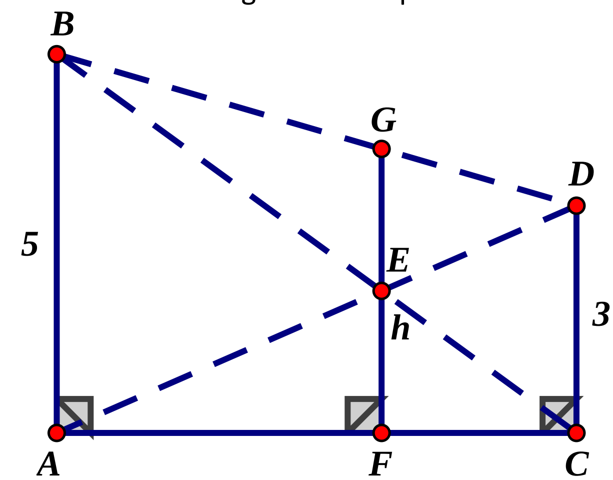
20. The diagonals of trapezoid *ABCD* intersect at *E*, and segment *FG* is parallel to the bases through *E*. *AB* = *a* and *CD* = *b*. Find the length *h* of segment *FG*, in terms of *a* and *b*.



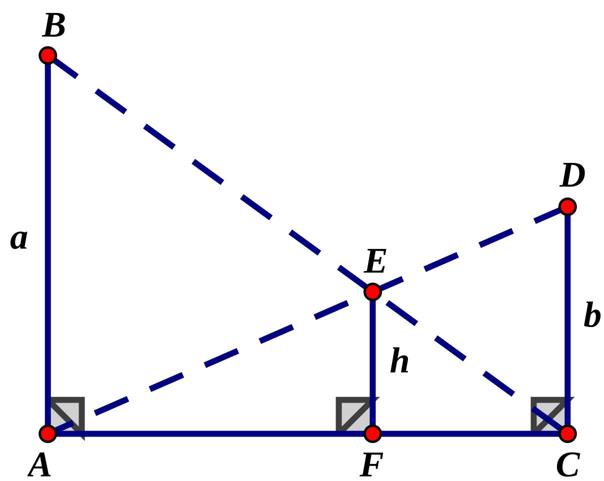
21. Pole *AB* is 5 feet tall, and pole *CD* is 3 feet tall. Wires from the top of each pole to the bottom of the other pole cross at point *E*. Find the height *h* of the point of intersection of the wires.



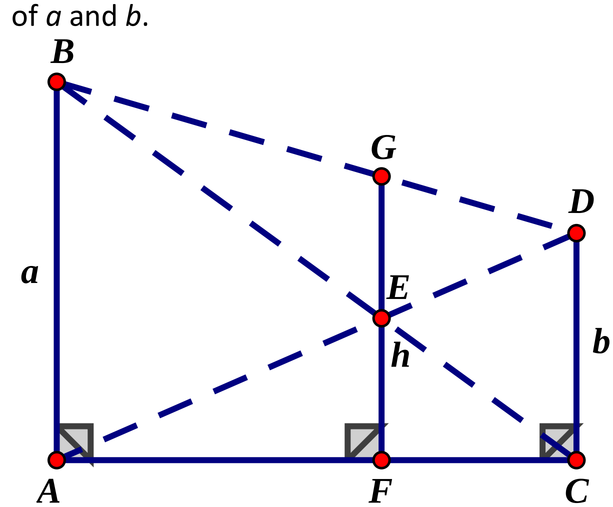
22. Pole *AB* is 5 feet tall, and pole *CD* is 3 feet tall. Wires from the top of each pole to the bottom of the other pole cross at point *E*. A pole through *E* touches the wire connecting the top of the poles at *G*. Find the height *h* of the pole *FG*.



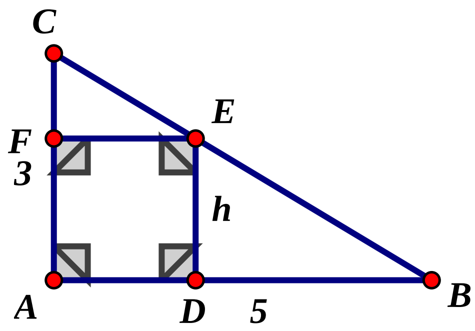
23. Pole *AB* is *a* feet tall, and pole *CD* is *b* feet tall. Wires from the top of each pole to the bottom of the other pole cross at point *E*. Find the height *h* of the point of intersection of the wires, in terms of *a* and *b*.



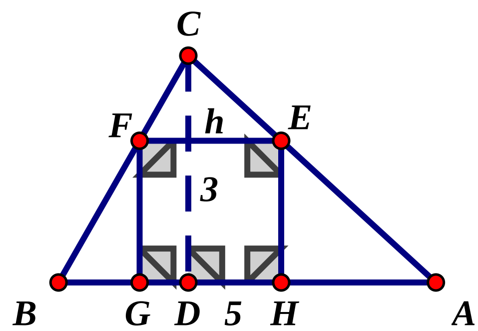
24. Pole *AB* is a feet tall, and pole *CD* is b feet tall. Wires from the top of each pole to the bottom of the other pole cross at point *E*. A pole through *E* touches the wire connecting the top of the poles at *G*. Find the height *h* of the pole *FG*, in terms of *a* and *b*.



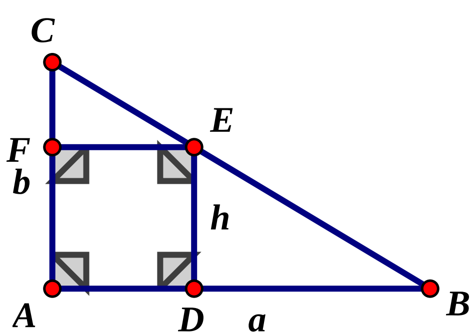
25. Square *ADEF* is inscribed in right triangle *ABC*. Side *AB* = 5 and *AC* = 3. Find the length *h* of the side of the square.



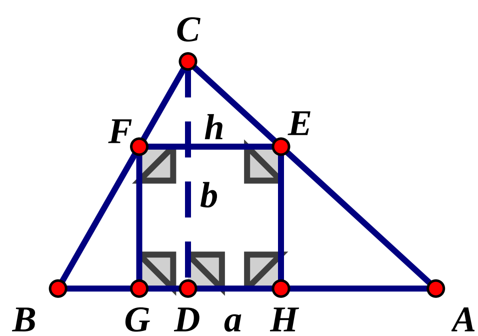
26. Triangle *ABC* has acute angles at *A* and *B*. Square *EFGH* is inscribed in triangle *ABC* with side *GH* corresponding to side *AB*. Side *AB* = 5 and the altitude *CD* to side *AB* has length 3. Find the length *h* of the side of the square.



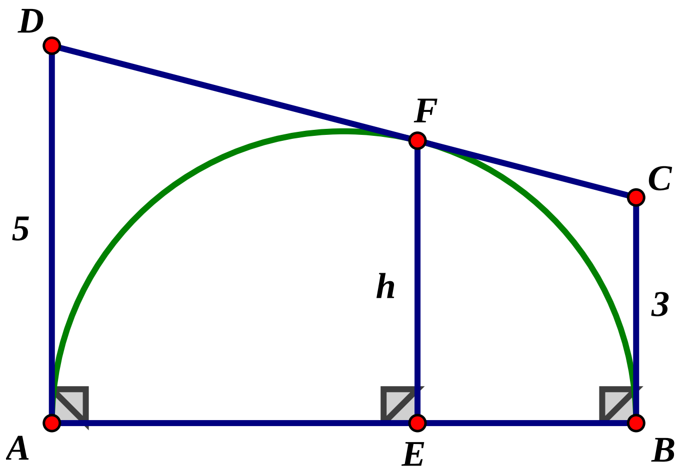
27. Square *ADEF* is inscribed in right triangle *ABC*. Side *AB* = *a* and *AC* = *b*. Find the length *h* of the side of the square, in terms of *a* and *b*.



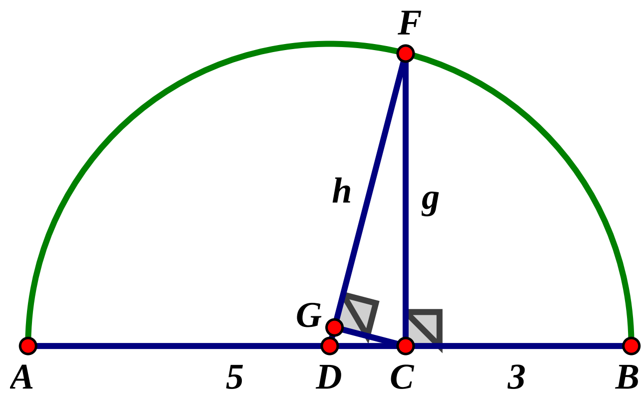
28. Triangle *ABC* has acute angles at *A* and *B*. Square *EFGH* is inscribed in triangle *ABC* with side *GH* corresponding to side *AB*. Side *AB* = *a* and the altitude *CD* to side *AB* has length *b*. Find the length *h* of the side of the square, in terms of *a* and *b*.



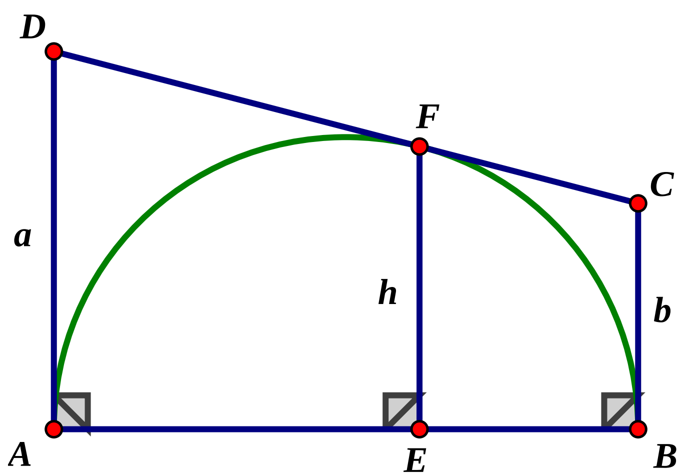
29. The semicircle with endpoints of the diameter at *A* and *B* is inscribed in trapezoid *ABCD* as shown. Base *AD* = 5 and base *BC* = 3. Find the length *h* of the perpendicular segment *EF* drawn from the point of tangency.



30. In semicircle *D*, the segment *AC* of the diameter has length 5, and the segment *BC* of the diameter has length 3. *CF* is perpendicular to the diameter at *C*, and *DF* is a radius. *CG* is perpendicular to the radius. Find the length *h* of segment *FG*.



31. The semicircle with endpoints of the diameter at *A* and *B* is inscribed in trapezoid *ABCD* as shown. Base *AD* = *a* and base *BC* = *b*. Find the length *h* of the perpendicular segment *EF* drawn from the point of tangency, in terms of *a* and *b*.



32. In semicircle *D*, the segment *AC* of the diameter has length *a*, and the segment *BC* of the diameter has length *b*. *CF* is perpendicular to the diameter at *C*, and *DF* is a radius. *CG* is perpendicular to the radius. Find the length *h* of segment *FG*, in terms of *a* and *b*.